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The reduction of graph families closed under contraction

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Abstract

Let $\mathcal{S}$ be a family of graphs. Suppose there is a nontrivial graph H such that for any supergraph G of H, G is in $\mathcal{S}$ if and only if the contraction G/H is in $\mathcal{S}$. Examples of such an $\mathcal{S}$: graphs with a spanning closed trail; graphs with at least k edge-disjoint spanning trees; and k-edge-connected graphs (k fixed). We give a reduction method using contractions to find when a given graph is in $\mathcal{S}$ and to study its structure if it is not in $\mathcal{S}$. This reduction method generalizes known special cases.

Keywords: Contraction; Spanning tree; Edge-arboricity; Edge-connectivity; Eulerian; Supereulerian

1. Introduction

We use the notation of Bondy and Murty [1], except that we do not allow graphs to have loops, we regard K_1 as k-edge-connected for all $k \in \mathrm{N}$, and we call a graph *trivial* if it is edgeless.

Let H (not necessarily connected) be a subgraph of G. The *contraction* G/H is the graph obtained from G by contracting all edges in H and by deleting any resulting loops. If $e \in E(G)$, then we denote $G/G[e]$ by G/e.

A collection $\mathcal{S}$ of graphs is called a *graph family* or a *family*. When G and H are graphs, if H is a subgraph of G, we denote this by $H \subseteq G$. Call a family $\mathcal{S}$ of graphs *closed under contraction* if

$$G \in \mathcal{S},\ e \in E(G) \Rightarrow G/e \in \mathcal{S}. \tag{1}$$

Call a family $\mathcal{C}$ of graphs *complete* if $\mathcal{C}$ satisfies these three axioms:

(C1) $\mathcal{C}$ contains all edgeless graphs;

(C2) $\mathcal{C}$ is closed under contraction;

(C3) $H \subseteq G,\ H \in \mathcal{C},\ G/H \in \mathcal{C} \Rightarrow G \in \mathcal{C}$.

[†] Sadly, the author passed away on April 20, 1995.

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Call a family $\mathscr{F}$ of graphs *free* if these three axioms hold:

(F1) $\mathscr{F}$ contains all edgeless graphs;

(F2) $G \in \mathscr{F},\ H \subseteq G \Rightarrow H \in \mathscr{F}$;

(F3) For any induced subgraph H of G,

$$H \in \mathscr{F} \quad \text{and} \quad G/H \in \mathscr{F} \Rightarrow G \in \mathscr{F}.$$

For any family $\mathscr{S}$ of graphs, we define the *kernel* $\mathscr{S}^{\mathrm{O}}$ of $\mathscr{S}$ to be the family

$$\mathscr{S}^{\mathrm{O}} = \{H \mid \text{for every supergraph } G \text{ of } H, G \in \mathscr{S} \Leftrightarrow G/H \in \mathscr{S}\}. \tag{2}$$

Obviously, $\mathscr{S}^{\mathrm{O}}$ contains all edgeless graphs. If $\mathscr{S}^{\mathrm{O}} = \{\text{edgeless graphs}\}$, then we call $\mathscr{S}^{\mathrm{O}}$ *trivial*.

Let $\mathscr{S}$ be a family $\mathscr{S}$ with a nontrivial kernel $\mathscr{S}^{\mathrm{O}}$ that is closed under contraction. Is a given graph G (say) in $\mathscr{S}$? Subgraphs of G in the kernel $\mathscr{S}^{\mathrm{O}}$ can each be contracted, and this can be repeated, until a 'reduced' graph G_1 (say) is obtained, having no nontrivial subgraph in $\mathscr{S}^{\mathrm{O}}$, where (2) implies

$$G \in \mathscr{S} \quad \text{if and only if} \quad G_1 \in \mathscr{S}. \tag{3}$$

By (3), to know if $G \in \mathscr{S}$ it suffices merely to know if the 'reduced' graph G_1 is in $\mathscr{S}$. If $\mathscr{S}^{\mathrm{O}}$ is nontrivial, then this can be easier than determining directly whether $G \in \mathscr{S}$. (We shall prove that this 'reduced graph' G_1 is uniquely determined by G and $\mathscr{S}^{\mathrm{O}}$, if $\mathscr{S}^{\mathrm{O}}$ is closed under contraction; that the family of all such 'reduced' graphs, corresponding to a given $\mathscr{S}$, is free; that if $\mathscr{S}$ or $\mathscr{S}^{\mathrm{O}}$ is closed under contraction, then $\mathscr{S}^{\mathrm{O}}$ is a complete family; that all complete families arise as kernels; and that all free families arise as families of 'reduced graphs'.)

For any family $\mathscr{T}$ of graphs, define

$$\mathscr{T}^{\mathrm{R}} = \{G \mid G \text{ has no nontrivial subgraph in } \mathscr{T}\} \tag{4}$$

and

$$\mathscr{T}^{\mathrm{C}} = \{G \mid G \text{ has no nontrivial contraction in } \mathscr{T}\}.$$

(This family $\mathscr{T}^{\mathrm{R}}$ is a family of 'reduced' graphs corresponding to $\mathscr{T}$, when $\mathscr{T}$ is a kernel. The family $\mathscr{T}^{\mathrm{C}}$ is the dual concept.) We shall also show that if $\mathscr{C}$ and $\mathscr{F}$ are families of graphs such that $\mathscr{C}^{\mathrm{R}} = \mathscr{F}$ and $\mathscr{F}^{\mathrm{C}} = \mathscr{C}$, then $\mathscr{C}$ is a complete family and $\mathscr{F}$ is a free family. Furthermore, all complete and free families arise this way.

2. Examples: complete families and kernels

Define the family $\mathscr{SL}$ of *supereulerian graphs*: $G \in \mathscr{SL}$ whenever G has a spanning closed trail, and K_1 is regarded as being in $\mathscr{SL}$. Thus, if $G \in \mathscr{SL}$ then G is the spanning supergraph of an eulerian graph, and K_1 is regarded as eulerian. Clearly, $\mathscr{SL}$ is closed under contraction. A graph G is called *collapsible* if for every even subset X of $V(G), G$ has a spanning connected subgraph H with X as its set of odd-degree vertices (see [2,3]). By Theorem 3 of [2] and its corollary, the family $\mathscr{CL}$ of graphs whose components are collapsible is a complete family, and $\mathscr{CL} \subseteq \mathscr{SL}^{\mathrm{O}}$. We conjecture that $\mathscr{CL} = \mathscr{SL}^{\mathrm{O}}$.

For any natural number k, let $\mathscr{C}(k)$ be the family of graphs with the property that for any $2k$ vertices $s_1, t_1, s_2, t_2, \ldots, s_k, t_k \in V(G)$ (not necessarily distinct) there are pairwise disjoint (s_i, t_i)-paths P_i $(1 \leqslant i \leqslant k)$. The family $\mathscr{C}(k)$ is easily shown to be complete, and its members are called *weakly k-linked.* Seymour [7] and Thomassen [8] have characterized $\mathscr{C}(2)$.

Lai [4] (and Theorem 4 of [5]) proved that if $\mathscr{S}$ is a complete family and if $\mathscr{C}_k$ is the family of graphs at most k edges short of being in $\mathscr{C}$, then $\mathscr{C}_k^{\mathrm{O}} = \mathscr{C}$.

3. Complete families and kernels

In the results of this section, $\mathscr{T}$, $\mathscr{S}$ and $\mathscr{C}$ will be various graph families, and $\mathscr{C}$ will often be complete. For the special case $\mathscr{S} = \mathscr{SL}$ and $\mathscr{C} = \mathscr{CL}$, some results below were first done in [2]: Theorem 4, Corollary 2 of Theorem 4, and Lemma 4 of [2] are generalized below to Theorem 3.7, Corollary 3.8, and Lemma 3.9, respectively.

Lemma 3.1. *Let $\mathscr{T}$ be a graph family. If*

$$\mathscr{T} \text{ contains all edgeless graphs,} \tag{5}$$

then $\mathscr{T}^{\mathrm{O}} \subseteq \mathscr{T}$.

Proof. Let $\mathscr{T}$ be a family satisfying (5) and suppose $G' \in \mathscr{T}^{\mathrm{O}}$. By (2),

$$G \in \mathscr{T} \Leftrightarrow G/G' \in \mathscr{T} \tag{6}$$

holds for every supergraph G of G'. Set $G = G'$ in (6) and use (5) to get $G' \in \mathscr{T}$. Hence, $\mathscr{T}^{\mathrm{O}} \subseteq \mathscr{T}$. □

Lemma 3.2. *If $\mathscr{S}$ is a graph family then $(\mathscr{S}^{\mathrm{O}})^{\mathrm{O}} = \mathscr{S}^{\mathrm{O}}$; also, all edgeless graph are in $\mathscr{S}$ if and only if $\mathscr{S}^{\mathrm{O}} \subseteq \mathscr{S}$.*

Proof. Let $\mathscr{S}$ be a graph family. Now, all edgeless graphs are in $\mathscr{S}^{\mathrm{O}}$, and so $\mathscr{S}^{\mathrm{O}} \subseteq \mathscr{S}$ implies that $\mathscr{S}$ contains all edgeless graphs. Set $\mathscr{S} = \mathscr{T}$ in Lemma 3.1 to get the last part of Lemma 3.2. Set $\mathscr{S}^{\mathrm{O}} = \mathscr{T}$ in Lemma 3.1 to get $(\mathscr{S}^{\mathrm{O}})^{\mathrm{O}} \subseteq \mathscr{S}^{\mathrm{O}}$. It remains to prove

$$\mathscr{S}^{\mathrm{O}} \subseteq (\mathscr{S}^{\mathrm{O}})^{\mathrm{O}}. \tag{7}$$

Let $H \in \mathscr{S}^{\mathrm{O}}$, let G' be a supergraph of H, and let G be an arbitrary supergraph of G'. Hence,

$$G/G' = (G/H)/(G'/H), \tag{8}$$

and since $H \in \mathscr{S}^O$, (2) implies

$$G/H \in \mathscr{S} \Leftrightarrow G \in \mathscr{S}. \tag{9}$$

If $G' \in \mathscr{S}^O$, then by (2),

$$G \in \mathscr{S} \Leftrightarrow G/G' \in \mathscr{S}, \tag{10}$$

and by (8)–(10),

$$G/H \in \mathscr{S} \Leftrightarrow G/G' \in \mathscr{S} \Leftrightarrow (G/H)/(G'/H) \in \mathscr{S}. \tag{11}$$

Since G/H can be any supergraph of G'/H, (11) implies $G'/H \in \mathscr{S}^O$.

Conversely, if $G' \notin \mathscr{S}^O$, then for some supergraph G of G',

$$G \in \mathscr{S} \not\Leftrightarrow G/G' \in \mathscr{S}, \tag{12}$$

and so by (9), (12), and (8),

$$G/H \in \mathscr{S} \not\Leftrightarrow (G/H)/(G'/H) \in \mathscr{S}. \tag{13}$$

Therefore, (2) implies that $G'/H \notin \mathscr{S}^O$.

By the last two paragraphs,

$$G' \in \mathscr{S}^O \Leftrightarrow G'/H \in \mathscr{S}^O,$$

when G' is an arbitrary supergraph of H. Hence, $H \in (\mathscr{S}^O)^O$, whence (2) implies (7). $\square$

Theorem 3.3. *For any graph family $\mathscr{S}$, if $\mathscr{S}$ or $\mathscr{S}^O$ is closed under contraction, then $\mathscr{S}^O$ is complete.*

Proof. Let $\mathscr{S}$ be a graph family.

First we show that $\mathscr{C} = \mathscr{S}^O$ satisfies (C1) and (C3). By Lemma 3.2, $(\mathscr{S}^O)^O = \mathscr{S}^O$. This and Lemma 3.2 imply that $\mathscr{S}^O$ satisfies (C1). Also, $(\mathscr{S}^O)^O = \mathscr{S}^O$ implies that $\mathscr{C} = \mathscr{S}^O$ satisfies (C3): for if $H \in \mathscr{S}^O$ and $H \subseteq G$ then $H \in (\mathscr{S}^O)^O$ and so (2) gives $G/H \in \mathscr{S}^O \Leftrightarrow G \in \mathscr{S}^O$.

By hypothesis, either $\mathscr{S}$ or $\mathscr{S}^O$ is closed under contraction. In the latter case $\mathscr{S}^O$ satisfies (C2), and so $\mathscr{S}^O$ is complete.

It only remains to suppose that $\mathscr{S}$ is closed under contraction and to prove that $\mathscr{S}^O$ is closed under contraction. Let $G \in \mathscr{S}^O$. For all supergraphs G' of G, (2) implies

$$G' \in \mathscr{S} \Leftrightarrow G'/G \in \mathscr{S}. \tag{14}$$

For any edge $e \in E(G)$, we have

$$(G'/e)/(G/e) = G'/G. \tag{15}$$

To prove that $\mathscr{S}^{\mathrm{O}}$ is closed under contraction, it suffices to prove $G/e \in \mathscr{S}^{\mathrm{O}}$, i.e., by (2), that

$$G'/e \in \mathscr{S} \Leftrightarrow (G'/e)/(G/e) \in \mathscr{S} \tag{16}$$

for all supergraphs G'/e of G/e. Let G' be any supergraph of G.

Suppose that $G' \in \mathscr{S}$. Since $\mathscr{S}$ is closed under contraction,

$$G'/e \in \mathscr{S} \tag{17}$$

and

$$G'/G \in \mathscr{S}. \tag{18}$$

By (18) and (15),

$$(G'/e)/(G/e) \in \mathscr{S}. \tag{19}$$

Suppose that $G' \notin \mathscr{S}$. By (14), we have $G'/G \notin \mathscr{S}$, and so by (15),

$$(G'/e)/(G/e) \notin \mathscr{S}. \tag{20}$$

By (20) and since $\mathscr{S}$ is closed under contraction,

$$G'/e \notin \mathscr{S}. \tag{21}$$

When $G' \in \mathscr{S}$, both (17) and (19) hold, but if $G' \notin \mathscr{S}$, then both (21) and (20) hold. Therefore, (16) holds, as claimed. □

Theorem 3.4. *For any family $\mathscr{C}$ of graphs that is closed under contraction, these are equivalent*:

(a) *$\mathscr{C}$ is the kernel of some graph family closed under contraction*;

(b) *$\mathscr{C}$ is a complete family*;

(c) $\mathscr{C} = \mathscr{C}^{\mathrm{O}}$.

Proof. (a) $\Rightarrow$ (b): By Theorem 3.3.

(b) $\Rightarrow$ (c): By (b), $\mathscr{C}$ is a complete family, and so (C1) and Lemma 3.1 give $\mathscr{C}^{\mathrm{O}} \subseteq \mathscr{C}$. Now suppose that $H \in \mathscr{C}$, and let G satisfy $H \subseteq G$. Since $\mathscr{C}$ is complete, $G/H \in \mathscr{C} \Leftrightarrow G \in \mathscr{C}$, because axiom (C2) implies '$\Leftarrow$' and axiom (C3) implies '$\Rightarrow$'. Hence, $H \in \mathscr{C}^{\mathrm{O}}$, and (c) follows.

(c) $\Rightarrow$ (a): If (c) holds, then $\mathscr{C}$ is the kernel of itself. □

Hong-Jian Lai (personal communication) has shown that part (a) of Theorem 3.4 can be replaced by '$\mathscr{C}$ is the kernel of some graph family that is both closed under contraction and not complete'.

Let $\mathscr{S}$ be the family of all connected graphs of odd order. Then $\mathscr{S} = \mathscr{S}^{\mathrm{O}}$, and since $\mathscr{S}$ is not closed under contraction, neither is $\mathscr{S}^{\mathrm{O}}$. Therefore, the kernel $\mathscr{S}^{\mathrm{O}}$ is not complete. Hence, in Theorems 3.3 and 3.4, we need the hypothesis of closure under contraction.

By (a) $\Leftrightarrow$ (c) of Theorem 3.4, any kernel $\mathscr{C}$ of a graph family closed under contraction satisfies (C2), and hence contains multigraphs of order 2. For practical purposes, to test whether a graph family $\mathscr{S}$ (closed under contraction) has a nontrivial kernel $\mathscr{S}^{\mathrm{O}}$, simply look for an order 2 multigraph H in $\mathscr{S}^{\mathrm{O}}$ of (2). This is generally easy to check.

A family $\mathscr{T}$ of graphs is called *closed under edge-addition* if for any graph G and edge $e \in E(G)$, $G - e \in \mathscr{T}$ implies $G \in \mathscr{T}$.

Theorem 3.5. *In any complete family, the subfamily of connected graphs is closed under edge-addition.*

Proof. Let $\mathscr{C}$ be the subfamily of connected graphs in a complete family, let G be a graph and let $e \in E(G)$. Suppose $G - e \in \mathscr{C}$. By (b)$\Rightarrow$(c) of Theorem 3.4, $G - e \in \mathscr{C}^{\mathrm{O}}$, and so $G \in \mathscr{C} \Leftrightarrow G/(G-e) \in \mathscr{C}$. Since $G - e$ is connected and $\mathscr{C}$ is complete, $G/(G-e) = K_1 \in \mathscr{C}$. Hence $G \in \mathscr{C}$. $\square$

Lemma 3.6. *If $\mathscr{C}$ is complete and $G \in \mathscr{C}$, then $G \cup K_1 \in \mathscr{C}$.*

Proof. Apply (C3) with $H \subseteq G$ of (C3) replaced by $G \subseteq G \cup K_1$. Then G/H of (C3) is an edgeless graph, and by (C1) it is in $\mathscr{C}$. $\square$

Theorem 3.7. *Let $\mathscr{C}$ be a complete family of graphs. Let H be a graph containing subgraphs H_1 and H_2, and satisfying*

$$H_1 \cup H_2 = H. \tag{22}$$

If H_1, $H_2 \in \mathscr{C}$, then $H \in \mathscr{C}$.

Proof. Let H be a graph with subgraphs H_1 and H_2 satisfying (22). Suppose that $\mathscr{C}$ is a complete graph family, and suppose H_1, $H_2 \in \mathscr{C}$.

The graph H/H_1 can be obtained from H_2 by a sequence of edge-additions, additions of isolated vertices, and contractions (contract newly added edges, to identify certain vertices of H_2 in H). Since $H_2 \in \mathscr{C}$ and since $\mathscr{C}$ is complete, $H/H_1 \in \mathscr{C}$, by (C2), by Theorem 3.5, and by Lemma 3.6.

Since $\mathscr{C}$ is complete, (b) $\Rightarrow$ (c) of Theorem 3.4 implies $H_1 \in \mathscr{C} = \mathscr{C}^{\mathrm{O}}$. Hence $H \in \mathscr{C}$, because (2) implies

$$H \in \mathscr{C} \Leftrightarrow H/H_1 \in \mathscr{C}. \qquad \square$$

Corollary 3.8. *Let $\mathscr{C}$ be a complete family and let G be a graph. Let E'' be a minimal edge set such that every component of $G - E''$ is in $\mathscr{C}$. Let E' be the edges of G that lie in no subgraph of G in $\mathscr{C}$. Then $E'' = E'$ and the set of maximal subgraphs of G in $\mathscr{C}$ is unique.*

Proof. If $e \in E(G) - E''$ then $e \notin E'$, and so $E' \subseteq E''$. By contradiction, suppose that there is an edge $xy \in E'' - E'$. Let H_x and H_y denote the components of $G - E''$ containing x and y, respectively. Thus, H_x, $H_y \in \mathscr{C}$. Since $xy \notin E'$, xy is in a subgraph H_{xy} (say) in $\mathscr{C}$. By Theorem 3.7, $H_x \cup H_{xy} \in \mathscr{C}$ and so $(H_x \cup H_{xy}) \cup H_y \in \mathscr{C}$. Therefore, each component of $G-(E''-E(H_{xy}))$ is in $\mathscr{C}$, contrary to the minimality of E''. Hence, E'' is uniquely determined. Since the maximal connected subgraphs of G in $\mathscr{C}$ are the components of $G - E''$, they are uniquely determined, too. □

Lemma 3.9. *Let $\mathscr{C}$ be a complete family, let G be a graph, and let H be a connected subgraph of G in $\mathscr{C}$. Let E'' be a minimal subset of $E(G)$ such that every component of $G-E''$ is in $\mathscr{C}$; let E^{**} be a minimal subset of $E(G/H)$ such that every component of $(G/H) - E^{**}$ is in $\mathscr{C}$; and let*

$$E' = \{e \in E(G) \mid e \text{ is in no subgraph of } G \text{ in } \mathscr{C}\}$$

and

$$E^* = \{e \in E(G/H) \mid e \text{ is in no subgraph of } G/H \text{ in } \mathscr{C}\}.$$

Then

$$E'' = E' = E^* = E^{**}. \tag{23}$$

Proof. The first and last equalities of (23) are instances of Corollary 3.8. It remains to prove $E' = E^*$.

Let H be a connected subgraph of G where $H \in \mathscr{C}$, let $e \in E'$, and suppose $e \notin E^*$, by way of contradiction. Then e is in a subgraph H'' of G/H where $H'' \in \mathscr{C}$. Denote by G'' the subgraph of G induced by $E(H) \cup E(H'')$. Thus,

$$H \subseteq G'', \quad H \in \mathscr{C}, \quad G''/H = H'' \in \mathscr{C},$$

and so by (C3), $G'' \in \mathscr{C}$. But, $e \in E(H'') \subseteq E(G'')$, contrary to $e \in E'$. Therefore,

$$E' \subseteq E^*. \tag{24}$$

Let $e \in E(G) - E'$. Hence by Corollary 3.8, G has a unique maximal subgraph $H_0 \in \mathscr{C}$ such that $e \in E(H_0)$. If H and H_0 are disjoint, then $e \in E(H_0)$, $H_0 \subseteq G/H$, and $H_0 \in \mathscr{C}$ jointly imply

$$e \notin E^*. \tag{25}$$

Since (25) holds whenever $e \notin E'$, (24) implies $E' = E^*$. □

Let $\mathscr{C} = \{C_3\}$ (not a complete family) and let G be the graph with $V(G) = \{a, b, c, d, e\}$ and

$$E(G) = \{ab,\ bc,\ cd,\ de,\ ea,\ ac,\ ce\}.$$

Now consider what happens if subgraphs in $\mathscr{C}$ (i.e., 3-cycles) are contracted until none remain. If $H = G[\{a, c, e\}]$ is contracted, then G/H has order 3 and no subgraph in

$\mathscr{C}$. If instead $H' = G[\{a,b,c\}]$ is contracted, then G/H' has a 3-cycle on $\{c,d,e\}$, and when the latter 3-cycle is also contracted, then only one vertex remains (which obviously has no subgraph in $\mathscr{C}$). This trivial graph is not isomorphic to G/H. We shall show next that if $\mathscr{C}$ is a complete family, then there is a unique graph having no subgraph in $\mathscr{C}$ that is obtained from G by any sequence of contractions of subgraphs in $\mathscr{C}$.

4. Free families and reduced graphs

Let $\mathscr{C}$ be a complete family and let G be a graph. By Corollary 3.8, G has a unique maximal spanning subgraph

$$G' = G - E'' = G - E'$$

(where E'' and E' are the sets of Corollary 3.8), with components in $\mathscr{C}$. Denote the components of G' by $\{H_1, H_2, \ldots, H_c\}$. Define the *$\mathscr{C}$-reduction* of G, called $G/\mathscr{C}$, to be the graph obtained from G by contracting each H_i $(1 \leqslant i \leqslant c)$ to a distinct vertex and by removing any resulting loops. If G has no nontrivial subgraph in $\mathscr{C}$, then $G = G/\mathscr{C}$, and we call G *$\mathscr{C}$-reduced*. For any family $\mathscr{S}$, and for any graph G, the $\mathscr{S}^{\mathrm{O}}$-reduction of G is K_1 if and only if G is in the kernel $\mathscr{S}^{\mathrm{O}}$ of $\mathscr{S}$.

Theorem 4.1. *If $\mathscr{C}$ is a complete family and G is a graph, then the $\mathscr{C}$-reduction of G, i.e. $G/\mathscr{C}$, is the unique $\mathscr{C}$-reduced graph obtained from G by contractions of subgraphs in $\mathscr{C}$.*

Proof. Let $\mathscr{C}$ be a complete family, let G be a graph, and let E'' and E' have the meaning of Lemma 3.9 (and of Corollary 3.8). Let G_1 be a reduced graph obtained from G by a sequence of contractions of connected subgraphs of G in $\mathscr{C}$. As G is contracted to G_1 by a sequence of contractions of connected subgraphs of G, Lemma 3.9 asserts that E'' and E' remain constant and equal throughout every step of the sequence. Since G_1 is $\mathscr{C}$-reduced, G_1 has no edge in any subgraph in $\mathscr{C}$, and so $E(G_1) \subseteq E'$. As G is contracted to G_1, the only edges that are contracted are edges in subgraphs in $\mathscr{C}$, and so the constancy of E' implies $E' \subseteq E(G_1)$. Hence, $E(G_1) = E' = E''$ and by definition, G_1 must be $G/\mathscr{C}$. □

For any complete family $\mathscr{C}$, the family $\mathscr{C}^{\mathrm{R}}$ (defined in (4)) is the family of $\mathscr{C}$-reduced graphs.

Corollary 4.2. *Let $\mathscr{C}'$ and $\mathscr{C}''$ be complete families of graphs. If $\mathscr{C}' \subseteq \mathscr{C}''$ then $(\mathscr{C}'')^{\mathrm{R}} \subseteq (\mathscr{C}')^{\mathrm{R}}$.*

Proof. If $G \in (\mathscr{C}'')^{\mathrm{R}}$, then G is $\mathscr{C}''$-reduced, and so $G = G/\mathscr{C}''$. By Theorem 4.1, $G/\mathscr{C}''$ has no nontrivial subgraph in $\mathscr{C}''$. Since $\mathscr{C}' \subseteq \mathscr{C}''$, $G/\mathscr{C}''$ thus has no nontrivial subgraph in $\mathscr{C}'$, and hence by definition, $G/\mathscr{C}''$ is $\mathscr{C}'$-reduced. Hence $G \in (\mathscr{C}')^{\mathrm{R}}$. □

There is a duality between complete families and free families, and between the operations $\mathscr{C} \to \mathscr{C}^R$ and $\mathscr{F} \to \mathscr{F}^C$, where $\mathscr{C}$ is complete and $\mathscr{F}$ is free. This duality appears below, and it has been studied further in [5]. For our purposes here, a contraction is trivial whenever it is edgeless, and any graph with an edge is a nontrivial contraction of itself.

Lemma 4.3. *For any family $\mathscr{C}$, if H is a subgraph of G and if $G \in \mathscr{C}^R$, then $H \in \mathscr{C}^R$.*

Proof. By the definition of $\mathscr{C}^R$, since $G \in \mathscr{C}^R$, G is $\mathscr{C}$-reduced. By definition, any subgraph H of G is $\mathscr{C}$-reduced, and hence $H \in \mathscr{C}^R$. □

Lemma 4.4. *For any family $\mathscr{C}$, any graph in $\mathscr{C} \cap \mathscr{C}^R$ is edgeless.*

Proof. If $H \in \mathscr{C}^R$, then by definition H has no nontrivial subgraph in $\mathscr{C}$. □

Lemma 4.5. *For any family $\mathscr{F}$, any graph in $\mathscr{F} \cap \mathscr{F}^C$ is edgeless.*

Proof. If $G \in \mathscr{F}^C$ then no nontrivial contraction of G is in $\mathscr{F}$. □

Theorem 4.6. *For any family $\mathscr{C}$ that is closed under contraction, $\mathscr{C}^R$ is a free family.*

Proof. We show that $\mathscr{C}^R$ satisfies (F1)–(F3). By definition, all edgeless graphs are in $\mathscr{C}^R$, so (F1) holds. By Lemma 4.3, $\mathscr{C}^R$ satisfies (F2).

Suppose by contradiction that (F3) fails for G and some nontrivial induced subgraph H of G. Thus, $H \in \mathscr{C}^R$, $G/H \in \mathscr{C}^R$, but $G \notin \mathscr{C}^R$, and hence G has a nontrivial subgraph $G' \in \mathscr{C}$.

First, suppose $V(G') \subseteq V(H)$. Since H is an induced subgraph, $G' \subseteq H$. Since $H \in \mathscr{C}^R$, Lemma 4.3 implies that $G' \in \mathscr{C}^R$, too. Thus, $G' \in \mathscr{C} \cap \mathscr{C}^R$, which is impossible by Lemma 4.4.

Therefore, $V(G') \nsubseteq V(H)$, and so $G'/(H \cap G')$ is nontrivial, where $G'/(H \cap G')$ denotes G' is $H \cap G'$ is edgeless. Since $\mathscr{C}$ is closed under contraction and $G' \in \mathscr{C}$, we have $G'/(H' \cap G) \in \mathscr{C}$. Thus, G/H has the nontrivial subgraph $G'/(H \cap G')$ in $\mathscr{C}$, contrary to $G/H \in \mathscr{C}^R$. Hence, (F3) holds for $\mathscr{C}^R$, and so $\mathscr{C}^R$ is free. □

Closure under contraction is needed in Theorem 4.6. Let $\mathscr{C}$ be the family of all graphs of odd order. Then $\mathscr{C}$ is not closed under contraction. Clearly, $K_2 \in \mathscr{C}^R$. Suppose that $\mathscr{C}^R$ is free. Then (F3) and $K_2 \in \mathscr{C}^R$ imply that $\mathscr{C}^R$ contains trees of all odd orders. So does $\mathscr{C}$. This violates Lemma 4.4.

Lemma 4.7. *Let $\mathscr{F}$ be a free family containing K_2 as a member. The subfamily of connected graphs in $\mathscr{F}^C$ is closed under edge-addition.*

Proof. Let $\mathscr{F}$ be a free family containing K_2 as a member, and let G be a nontrivial graph with a distinguished edge e such that $H = G - e$ is connected. By contradiction, suppose that $H \in \mathscr{F}^C$ and $G \notin \mathscr{F}^C$. Then G has a nontrivial contraction G_0 (say) in $\mathscr{F}$, but H has no nontrivial contraction in $\mathscr{F}$.

Case 1: *Suppose* $e \notin E(G_0)$. Let $G_0(e)$ denote the graph to which G is contracted when the edges of $(E(G)-E(G_0))-e$ are contracted. First suppose that $e \notin E(G_0(e))$. Then the contraction (in G) of the edges of $(E(G)-E(G_0))-e$ identifies the ends of e, and hence $G_0 = G_0(e)$ and this $G_0(e)$ is also a contraction of $H = G-e$. But then H has a nontrivial contraction G_0 in $\mathscr{F}$, a contradiction. Therefore, $e \in E(G_0(e))$, and G_0 is obtained from $G_0(e)$ by contracting e. If $G_0(e)$ has an edge e' parallel to e, then $G_0 \in \mathscr{F}$ could be obtained from H by contracting H to $G_0(e)-e$ and then by contracting e', but this would violate the fact that H has no nontrivial contraction in $\mathscr{F}$. Hence, $G_0(e)$ has no edge e' parallel to e, and so $G_0(e)[e]$, a K_2, is an induced subgraph of $G_0(e)$.

Since $\mathscr{F}$ is a free family, since $G_0(e)[e] = K_2 \in \mathscr{F}$, and since $G_0(e)/e = G_0 \in \mathscr{F}$, (F3) implies that $G_0(e) \in \mathscr{F}$. By (F2), $G_0(e)-e \in \mathscr{F}$. Since $G-e$ is connected, so is $G_0(e)-e$, and it is nontrivial. Hence, $H = G-e$ has the nontrivial contraction $G_0(e)-e \in \mathscr{F}$, a contradiction precluding Case 1.

Case 2: *Suppose* $e \in E(G_0)$. By $G_0 \in \mathscr{F}$ and by (F2), $G_0 - e \in \mathscr{F}$. Since $G-e$ is connected, so is G_0-e, and so G_0-e is a nontrivial contraction of H lying in $\mathscr{F}$, contrary to $H \in \mathscr{F}^C$. □

Lemma 4.8. *For any family* $\mathscr{F}$, $\mathscr{F}^C$ *is closed under contraction.*

Proof. Let $\mathscr{F}$ be a family. If all members of $\mathscr{F}^C$ are edgeless, then the lemma is easy.

Suppose that $G \in \mathscr{F}^C$ and that G_0 is a nontrivial contraction of G. By the definition of $\mathscr{F}^C$, G has no nontrivial contraction in $\mathscr{F}$, and so neither does G_0. Thus, $G_0 \in \mathscr{F}^C$. □

Lemma 4.9. *If* $\mathscr{F}$ *is free and* $G \in \mathscr{F}$, *then* $G \cup K_1 \in \mathscr{F}$.

Proof. Apply (F3) with H and G, respectively, of (F3) replaced by G and $G \cup K_1$, respectively. Then G/H of (F3) is edgeless, and by (F1) it is in $\mathscr{F}$. □

Theorem 4.10. *Suppose* $\mathscr{F}$ *is a free family. Then the family* $\mathscr{C} = \mathscr{F}^C$ *is complete. Also,* $\mathscr{F} = \mathscr{C}^R = (\mathscr{F}^C)^R$.

Proof. If no graph in $\mathscr{F}$ has an edge, then $\mathscr{F}$ is the family of all edgeless graphs, $\mathscr{C} = \mathscr{F}^C$ is the family of all graphs, which is complete, and $\mathscr{C}^R$ is the family of all edgeless graphs.

Suppose that $\mathscr{F}$ is a free family such that some graph of $\mathscr{F}$ has an edge, and let $\mathscr{C} = \mathscr{F}^C$. By (F2), $K_2 \in \mathscr{F}$, so Lemma 4.7 applies. We must prove that $\mathscr{C}$ satisfies axioms (C1)–(C3) of the definition of a complete family, and that $\mathscr{F} = \mathscr{C}^R$. By definition, $\mathscr{C}$ satisfies (C1). By Lemma 4.8, (C2) holds.

We prove (C3). Let G be a supergraph of a nontrivial graph

$$H \in \mathscr{C}. \tag{26}$$

We claim

$$G/H \in \mathscr{C} \Rightarrow G \in \mathscr{C}. \tag{27}$$

By way of contradiction, suppose (27) is false. Then

$$G \notin \mathscr{C} \quad \text{and} \quad G/H \in \mathscr{C}. \tag{28}$$

By the definition of $\mathscr{C}$, $G \notin \mathscr{C}$ of (28) implies that G has a nontrivial contraction G_0 (say) in $\mathscr{F}$. Let $\theta : V(G) \to V(G_0)$ denote the surjection induced by this contraction. We claim first that there is an edge $e \in E(H) \cap E(G_0)$: otherwise, G/H can be contracted to the nontrivial graph $G_0 \in \mathscr{F}$, contrary to $G/H \in \mathscr{C} = \mathscr{F}^{\mathrm{C}}$ in (28). Let H_e be the component of H containing e. Denote

$$E = \{xy \mid \text{there is an } i \text{ such that } x, y \in \theta^{-1}(v_i) \cap H_e\}.$$

Let $J = (H/E)/(H - E(H_e))$. Note $J \in \mathscr{F}^{\mathrm{C}}$. Let H_0 be the subgraph of G_0 containing the edges of $H_e \cap G_0$ and no isolated vertices. Note that $H_0 \in \mathscr{F}$. Add enough isolated vertices to H_0 so that it will equal J. By Lemma 4.9, $J \in \mathscr{F}$, contradicting Lemma 4.5. This contradiction proves (27) and hence that $\mathscr{C}$ satisfies (C3).

Now we prove $\mathscr{F} \subseteq \mathscr{C}^{\mathrm{R}}$. Suppose $G \in \mathscr{F}$. By contradiction, if $G \notin \mathscr{C}^{\mathrm{R}}$ then G has a nontrivial subgraph $H \in \mathscr{C} = \mathscr{F}^{\mathrm{C}}$. By $G \in \mathscr{F}$ and (F2), $H \in \mathscr{F}$, and so by Lemma 4.5, H is trivial, a contradiction.

To prove $\mathscr{C}^{\mathrm{R}} \subseteq \mathscr{F}$, we suppose (by contradiction) that G is a minimal member of $\mathscr{C}^{\mathrm{R}} - \mathscr{F}$. Since $\mathscr{F}$ contains all edgeless graphs, G is a nontrivial graph in $\mathscr{C}^{\mathrm{R}}$. By Lemma 4.4, $G \notin \mathscr{C} = \mathscr{F}^{\mathrm{C}}$. One of these two cases holds:

Case A: *Suppose G is disconnected.* Let H be a component of G and let $H' = G - H$. By the minimality of G, both H and H' are in $\mathscr{F}$. Let G' denote the graph obtained by adding an edge e (say) joining some vertex of $V(H)$ and some vertex $V(H')$. Therefore, G' has vertex-induced subgraphs $G'[e]$, H, and H', all in $\mathscr{F}$ since $K_2 \in \mathscr{F}$. By two applications of (F3), $G' \in \mathscr{F}$. By (F2), $G = G' - e \in \mathscr{F}$, a contradiction.

Case B: *Suppose G is connected.* Since $G \notin \mathscr{F}^{\mathrm{C}}$, some nontrivial contraction G_0 (say) of G is in $\mathscr{F}$. Since $G \notin \mathscr{F}$, $G \neq G_0$. Since G is connected and $G_0 \neq K_1$, we have $E(G_0) \neq \emptyset$. Hence, $G - E(G_0)$ has $|V(G_0)| = c$ components, say $H_1, H_2, \ldots, H_c$, for some $c \geqslant 2$. Each H_i is an induced subgraph of G, and by Lemma 4.3, $H_i \in \mathscr{C}^{\mathrm{R}}$ $(1 \leqslant i \leqslant c)$. Since G was chosen to be a minimal member of $\mathscr{C}^{\mathrm{R}} - \mathscr{F}$ and since $c \geqslant 2$, each H_i $(1 \leqslant i \leqslant c)$ is in $\mathscr{F}$. But also $G_0 \in \mathscr{F}$, and so by repeated applications of axiom (F3), $G \in \mathscr{F}$. This contradiction proves $\mathscr{C}^{\mathrm{R}} = \mathscr{F}$, as claimed. □

In Theorem 4.10, $\mathscr{F}$ cannot be just any family. Suppose, for example, that $\mathscr{F}$ is the family of connected graphs of odd order. Thus, $\mathscr{F}$ violates (F2), so $\mathscr{F}$ is not a free family. It is easily seen that $\mathscr{F}^{\mathrm{C}}$ is not complete: $\mathscr{F}^{\mathrm{C}}$ contains K_2, and hence if (C3) held then $\mathscr{F}^{\mathrm{C}}$ would contain all trees. But trees of odd order are in $\mathscr{F}$, and Lemma 4.5 is violated.

Theorem 4.11. *If $\mathscr{C}$ is a complete family, then $(\mathscr{C}^{\mathrm{R}})^{\mathrm{C}} = \mathscr{C}$.*

Proof. Suppose that $\mathscr{C}$ is complete and let $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$. First suppose $G \in (\mathscr{C}^{\mathrm{R}})^{\mathrm{C}}$. By the definition of $\mathscr{F}^{\mathrm{C}}$, no nontrivial contraction H of G is in $\mathscr{C}^{\mathrm{R}}$. But by Theorem 4.1, the graph $G/\mathscr{C}$ is a contraction of G in $\mathscr{C}^{\mathrm{R}}$. Hence, $G/\mathscr{C}$ must be edgeless, and this implies that the components of G are in $\mathscr{C}$. Hence by Theorem 3.7, $G \in \mathscr{C}$, and so $(\mathscr{C}^{\mathrm{R}})^{\mathrm{C}} \subseteq \mathscr{C}$.

Suppose instead that $G \in \mathscr{C}$. The complete family $\mathscr{C}$ is closed under contraction and hence all contractions of G are in $\mathscr{C}$. Thus, by Lemma 4.4, G has no nontrivial contraction in $\mathscr{C}^{\mathrm{R}}$, and so by the definition of $\mathscr{F}^{\mathrm{C}}$, $G \in (\mathscr{C}^{\mathrm{R}})^{\mathrm{C}}$. Thus, $\mathscr{C} \subseteq (\mathscr{C}^{\mathrm{R}})^{\mathrm{C}}$. □

Theorem 4.12. *Let $\mathscr{C}$ and $\mathscr{F}$ be two graph families. If both $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$ and $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$, then $\mathscr{C}$ is a complete family and $\mathscr{F}$ is a free family. For any complete family $\mathscr{C}$ there is a free family $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$ such that $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$. For any free family $\mathscr{F}$ there is a complete family $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$ such that $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$.*

Proof. Let $\mathscr{C}$ and $\mathscr{F}$ be two graph families, and suppose $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$ and $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$. By Lemma 4.8, $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$ is closed under contraction. Hence, by Theorem 4.6, $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$ is a free family, and so by Theorem 4.10, $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$ is a complete family.

For any complete family $\mathscr{C}$, apply Theorems 4.6 and 4.11 to obtain the desired free family $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$. For any free family $\mathscr{F}$, apply Theorem 4.10 to obtain the desired complete family $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$. □

For the operations $\mathscr{C} \to \mathscr{C}^{\mathrm{R}}$ and $\mathscr{F} \to \mathscr{F}^{\mathrm{C}}$, it is natural to ask when families $\mathscr{C}$ and $\mathscr{F}$ exist satisfying $\mathscr{C} = \mathscr{F}^{\mathrm{C}}$ and $\mathscr{F} = \mathscr{C}^{\mathrm{R}}$. Thus, Theorem 4.12 motivates the study of complete families and free families. Our original motivation for considering these families was the study of the kernel $\mathscr{S}^{\mathrm{O}}$ and the corresponding reduced graphs, but Theorem 4.12 is another justification.

Theorem 4.13. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be free families of graphs. Then*

$$\mathscr{F}_1 \subseteq \mathscr{F}_2 \quad \textit{if and only if} \quad \mathscr{F}_2^{\mathrm{C}} \subseteq \mathscr{F}_1^{\mathrm{C}}.$$

Proof. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be free families. Suppose $\mathscr{F}_1 \subseteq \mathscr{F}_2$ and let $G \in \mathscr{F}_2^{\mathrm{C}}$. By definition, no nontrivial contraction of G is in $\mathscr{F}_2$. Hence, no nontrivial contraction of G is in $\mathscr{F}_1$, and so by definition, $G \in \mathscr{F}_1^{\mathrm{C}}$.

Conversely, suppose $\mathscr{F}_2^{\mathrm{C}} \subseteq \mathscr{F}_1^{\mathrm{C}}$. By Theorem 4.10, $\mathscr{F}_2^{\mathrm{C}}$ and $\mathscr{F}_1^{\mathrm{C}}$ are complete families. By Theorem 4.10 (twice) and Corollary 4.2,

$$\mathscr{F}_1 = (\mathscr{F}_1^{\mathrm{C}})^{\mathrm{R}} \subseteq (\mathscr{F}_2^{\mathrm{C}})^{\mathrm{R}} = \mathscr{F}_2. \qquad \square$$

Corollary 4.14. *Let $\mathscr{C}'$ and $\mathscr{C}''$ be complete families. Then*

$$\mathscr{C}' \subseteq \mathscr{C}'' \quad \textit{if and only if} \quad (\mathscr{C}'')^{\mathrm{R}} \subseteq (\mathscr{C}')^{\mathrm{R}}.$$

Proof. By Theorem 4.6, $\mathscr{F}_1 = (\mathscr{C}'')^{\mathrm{R}}$ and $\mathscr{F}_2 = (\mathscr{C}')^{\mathrm{R}}$ are free families. This and Theorem 4.11 imply both $\mathscr{F}_1^{\mathrm{C}} = ((\mathscr{C}'')^{\mathrm{R}})^{\mathrm{C}} = \mathscr{C}''$ and $\mathscr{F}_2^{\mathrm{C}} = ((\mathscr{C}')^{\mathrm{R}})^{\mathrm{C}} = \mathscr{C}'$. Applying Theorem 4.13, we get the result. □

5. Examples: free families

The smallest free family $\mathscr{F}$ containing a nontrivial graph is the family of all forests. (By (F2), if a free family $\mathscr{F}$ has any member with an edge, then $K_2 \in \mathscr{F}$. This and (F1) and (F3) imply that $\mathscr{F}$ contains all forests.) The corresponding complete family $\mathscr{F}^{\mathrm{C}}$ consists of all graphs with no cut-edges.

Corresponding to edge-connectivity $\kappa'(G)$, define

$$\overline{\kappa'}(G) = \max_{H \subseteq G} \kappa'(H).$$

Let $k \in \mathbf{N}$. If $\mathscr{C}$ is the complete family of graphs with k-edge-connected components, then $\mathscr{C}^{\mathrm{R}} = \{G \mid \overline{\kappa'}(G) < k\}$ is the corresponding free family.

For $k \geqslant 2$, define $\mathscr{F}_k = \{G \mid G \text{ has girth at least } k\}$. Then $\mathscr{F}_k$ is a free family, $\mathscr{F}_2$ is the family of all graphs, and $\mathscr{F}_3$ is the family of all simple graphs.

Define, for any nontrivial graph G,

$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1},$$

where the maximum runs over all nontrivial subgraphs H of G. Nash-Williams [6] showed that $\lceil \gamma(G) \rceil$, called the *edge-arboricity* of G, is the minimum number of forests whose union contains G. For $k \in \mathbf{N}$, the family of graphs with edge-arboricity at most k is a free family. If $\mathscr{C}$ is the complete family of graphs with k edge-disjoint spanning trees, then $\mathscr{C}^{\mathrm{R}}$ is the family of graphs G with edge-arboricity at most k, but with no nontrivial subgraph of G having k edge-disjoint spanning trees.

Suppose a free family $\mathscr{F}$ contains a graph having an n-cycle. By (F2), $K_2, C_n \in \mathscr{F}$. This and repeated applications of (F3) imply that all cycles of length at least n are in $\mathscr{F}$. For example, the free families $\mathscr{C}\mathscr{L}^{\mathrm{R}}$ and $(\mathscr{S}\mathscr{L}^{\mathrm{O}})^{\mathrm{R}}$ contain all cycles of length at least 4.

The complete family of graphs whose components all have two edge-disjoint spanning trees is contained (by Theorem 2 and the corollary of Theorem 3 of [2]) in the kernel $\mathscr{S}\mathscr{L}^{\mathrm{O}}$, a complete family, by Theorem 3.3. Hence, by Corollary 4.14, any graph G in $(\mathscr{S}\mathscr{L}^{\mathrm{O}})^{\mathrm{R}}$ has edge-arboricity at most 2.

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